

W-like states are not Necessary for Totally Correct Quantum Anonymous Leader Election

Alexander Norton

*School of Computer Science, McGill University
3480 rue University, Suite 318, Montreal, Quebec, H3A 2A7, Canada*

I show that W -like entangled quantum states are not a necessary quantum resource for totally correct anonymous leader election protocols. This is proven by defining a symmetric quantum state that is n -partite SLOCC inequivalent to the W state, and then constructing a totally correct anonymous leader election protocol using this state. This result, which contradicts the previous necessity result of D'Hondt and Panangaden, furthers our understanding of how non- local quantum states can be used as a resource for distributed computation.

1 Introduction

Leader election is a fundamental problem of distributed computing. The goal of a leader election protocol is to choose a single processor as a leader out of a network of eligible candidates. On an anonymous network where each processor has identical local information and therefore no way of being uniquely identified, the essence of the problem is to somehow break the symmetry between the processors. Constructing a leader election protocol that is guaranteed to terminate regardless of the network topology (a so-called *totally correct* protocol) is known to be impossible with only classical resources and classical communication [1]. However, in 2006 it was shown that pre-sharing certain quantum states across a network would enable totally correct anonymous leader election (TCAL) [2]. Furthermore, in 2012 a quantum TCAL protocol was devised that does not require pre-sharing of quantum resources, which proved that certain classically unsolvable problems can be solved with quantum communication and quantum computation [3].

In an attempt to classify exactly what kind of quantum resources are required for TCAL, D'Hondt and Panangaden provided a proof that on a network where only classical broadcast communication and local quantum operations on a pre-shared quantum resource are allowed, it is necessary and sufficient to share W -like states, which are multi-qubit extensions of the maximally entangled W state [2]. The main result of this paper is that the result of D'Hondt and Panangaden is incorrect. This is proved by defining a quantum state \tilde{W} that is symmetric, and therefore suitable for use in an anonymous distributed protocol [2]. Symmetric states are a class of quantum states whose properties with respect to the well-known SLOCC hierarchy of quantum states have been well studied [4, 5], and I use these properties to show that the \tilde{W} state is SLOCC inequivalent to the W state in the n -partite case. Then, a TCAL protocol relying only on the \tilde{W} state is provided, implying that W -like states are not necessary for totally correct anonymous leader election. As leader election protocols enable the efficient implementation of a variety of other fundamental distributed protocols [6], this result improves our understanding of the properties of non-local quantum states that allow them to be used

as resources for distributed computation.

2 Background

The following concepts will be used in the analysis of the \tilde{W} state.

2.1 SLOCC Hierarchy

A useful tool in the analysis of quantum resources is the Stochastic Local Operations with Classical Communication (SLOCC) hierarchy. Two states are of the same SLOCC class if there is a strictly positive probability of converting one to the other by means of only local quantum operations and classical communication. Mathematically, this is equivalent to the existence of invertible local operations which transform the first state into the second when applied across each subspace [7].

For states with fewer than 4 qubits there are a finite number of SLOCC classes, but the hierarchy becomes infinite beyond that point [7]. There are exactly two types of maximally entangled 3-partite states, with the 3-partite *GHZ* state $(1/\sqrt{2})(|000\rangle + |111\rangle)$ and the 3-partite *W* state $(1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$ being representative members respectively. Although both are entangled, their entanglement has different properties: Tracing out over any qubit of the *GHZ* state leaves a separable mixed state, whereas tracing out over any qubit of the *W* state leaves some entanglement between the remaining qubits.

The SLOCC hierarchy has been the subject of much investigation in recent history, with classification results existing for 4 qubits [8], n -qubit symmetric states [5], and finally the general n -partite case [9].

2.2 Symmetric State Representations

A quantum state $|\psi\rangle$ is *symmetric* if it is identical under all permutations of its subsystems. Symmetric states are required for a distributed quantum algorithm to be anonymous [2].

Define the permutation operator \mathcal{P} to be the sum over all qubit permutations of a given state. A natural way of representing any symmetric state $|\psi\rangle$ is as a superposition of the so-called Dicke states [5]:

$$|\psi\rangle = \sum_{k=0}^n \alpha_k \mathcal{P}(\underbrace{|0\dots 0\rangle}_k \underbrace{|1\dots 1\rangle}_{n-k}). \quad (1)$$

$|\psi\rangle$ can also be represented as the superposition of all qubit permutations of some specific quantum state. That is, for some qubits $|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle$

$$|\psi\rangle = \mathcal{N} \cdot \mathcal{P}(|\phi_1\rangle|\phi_2\rangle \dots |\phi_n\rangle), \quad (2)$$

where \mathcal{N} is a normalization factor. This is known as the Majorana representation. It is non-trivial that this representation always exists for symmetric states, and there is an algebraic relationship between the ϕ terms of the Majorana representation and the α weights of the Dicke states [5].

Let $\Phi = \{|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle\}$ be the Majorana terms of some symmetric state $|\psi\rangle$. The *degeneracy configuration* of the Majorana representation is the monotonically decreasing sequence of the cardinalities of the partitioning subsets of Φ that arise when grouping the ϕ terms by equality. There is a strong connection between the degeneracy configuration of two symmetric states and their SLOCC equivalence - if two states have different degeneracy configurations then they belong to different SLOCC classes [5].

2.3 W-like States

The n -partite W state is defined as

$$\begin{aligned} W_n &= \frac{1}{\sqrt{n}} \mathcal{P} |1 \underbrace{0 \dots 0}_{n-1}\rangle \\ &= \frac{1}{\sqrt{n}} (|10 \dots 0\rangle + |01 \dots 0\rangle + \dots + |00 \dots 1\rangle). \end{aligned} \quad (3)$$

Note that the W_n state is a symmetric state that is naturally represented as a single permutation term, and so its Majorana representation is trivially $\phi_1 = |1\rangle$ and $\phi_{2 \leq i \leq n} = |0\rangle$ which implies a degeneracy configuration of $n - 1, 1$.

From a computational perspective, a defining property of the W state is that if shared between n parties, when parties measure in the computational basis it is guaranteed that exactly one will measure $|1\rangle$ and all other parties will measure $|0\rangle$. It is these asymmetrical measurement results that allow W states to be used for leader election [2].

When considering multi-qubit extensions of this state for use in leader election protocols, it was this key property that was preserved by D'Hondt and Panangaden. They define W -like states by initially splitting the space of measurement results into those that will make a processor the leader and those that will not, and then constructing a permutation term as above [2]. Their result is that pre-shared W -like states are required for TCALE on networks where only SLOCC transformations are allowed. This will subsequently be shown to be incorrect.

3 SLOCC Inequivalence of W and \tilde{W}

Define the n -partite \tilde{W}_n state as the equal superposition of W_n and \overline{W}_n :

$$\tilde{W}_n = \frac{1}{\sqrt{2n}} (\mathcal{P} |1 \underbrace{0 \dots 0}_{n-1}\rangle + \mathcal{P} |0 \underbrace{1 \dots 1}_{n-1}\rangle). \quad (4)$$

First I show that \tilde{W}_n is SLOCC inequivalent from W_n for all $n > 2$ by means of the degeneracy configuration of its Majorana representation. This result implies that \tilde{W}_n is not a W -like state.

3.1 Majorana representation of \tilde{W}

Let \mathcal{R}_n be the set of the j th roots of the following polynomial, where $j = n - 2$, $n > 2$:

$$x^{n-2} + 1 = 0 \quad n \text{ is even} \quad (5)$$

$$x^{n-2} - 1 = 0 \quad n \text{ is odd.} \quad (6)$$

Explicitly,

$$\mathcal{R}_n = \begin{cases} \exp\left(\frac{(2k-1)\pi i}{n-2}\right) & 1 \leq k \leq n-2, n \text{ even} \\ \exp\left(\frac{2k\pi i}{n-2}\right) & 1 \leq k \leq n-2, n \text{ odd.} \end{cases} \quad (7)$$

Define

$$\mathcal{U}_n = \left\{ (1/\sqrt{2})(|0\rangle + r|1\rangle) \mid r \in \mathcal{R}_n \right\}, \quad (8)$$

$$\Phi_n = \mathcal{U}_n \cup \{|0\rangle, |1\rangle\} \equiv |\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle, \text{ and} \quad (9)$$

$$\mathcal{M}_n = \sum_{p \in \text{Perm}(n)} |\phi_{p(1)}\rangle \otimes |\phi_{p(2)}\rangle \otimes \dots \otimes |\phi_{p(n)}\rangle. \quad (10)$$

Lemma 1. Φ_n constitutes the Majorana representation of \tilde{W}_n for all $n > 2$.

Proof. By the definition of the Majorana representation [5] it is sufficient to show for all n -qubit computational basis vectors $|v\rangle$ that $\langle v|M_n\rangle = \mathcal{N} \cdot \langle v|\tilde{W}_n\rangle$ for some scalar \mathcal{N} independent of $|v\rangle$, as this immediately implies that $\tilde{W}_n = \mathcal{N} \cdot \mathcal{M}_n$ for normalization factor \mathcal{N} . Given $a_i \in \{0, 1\}$ $1 \leq i \leq n$ let $|v\rangle = |a_n a_{n-1} \dots a_1\rangle$ be the corresponding computational basis vector of an n -qubit system. Observe that $\langle v|\mathcal{M}_n\rangle =$

$$\sum_{p \in \text{Perm}(n)} \langle a_1|\phi_{p(1)}\rangle \times \langle a_2|\phi_{p(2)}\rangle \times \dots \times \langle a_n|\phi_{p(n)}\rangle. \quad (11)$$

The result of this expression depends on which of the $\langle a_i|$ are $\langle 0|$ and which are $\langle 1|$. The *Hamming weight* of $|v\rangle$ is defined as the number of $|a_i\rangle$ equal to $|1\rangle$. The sum over all permutations implies that for all computational basis vectors $|v\rangle, |u\rangle$ with equal Hamming weight, $\langle v|M_n\rangle = \langle u|M_n\rangle$. Let h be the Hamming weight of $|v\rangle$. As $\{|0\rangle, |1\rangle\} \subset \Phi_n$, when $h = 0$ or $h = n$ clearly

$$\langle v|M_n\rangle = 0. \quad (12)$$

$\langle v|\mathcal{M}_n\rangle$ is calculated for $2 \leq h \leq n-1$ by considering the permutations p that contribute non-zero terms to the sum. Let x_p be the term corresponding to permutation p . $x_p \neq 0$ implies that of the members of Φ_n , $|0\rangle$ is paired with $\langle 0|$ and $|1\rangle$ is paired with $\langle 1|$. The members of \mathcal{U}_n are partitioned between the $h-1$ remaining $\langle 1|$ s and the $n-h-1$ remaining $\langle 0|$ s, which implies that there will be exactly $h!(n-h)!$ terms with value x_p in the overall sum. x_p can be computed as follows. There are $h-1$ factors of the form $\langle 1|u\rangle$ and $n-h-1$ factors of the form $\langle 0|u'\rangle$, where $u, u' \in \mathcal{U}_n$. Furthermore each u appears in exactly one factor. Thus $x_p = (1/\sqrt{2})^{n-2} y_p$ where y_p is the product of $h-1$ specific members of \mathcal{R}_n . As the sum is over all permutations, each possible product of $h-1$ members of \mathcal{R}_n appears in the sum as one of the y_p . Defining \mathcal{G}_k^n to be the sum of all distinct products of k members of \mathcal{R}_n , the final result is written as

$$\langle v|\mathcal{M}_n\rangle = h!(n-h)!(1/\sqrt{2})^{n-2} \mathcal{G}_{h-1}^n. \quad (13)$$

By a very similar calculation to the above, when $h = 1$

$$\langle v|\mathcal{M}_n\rangle = (n-1)!(1/\sqrt{2})^{n-2}. \quad (14)$$

By definition \mathcal{R}_n is the set of the roots of the polynomial

$$\begin{aligned} x^{n-2} + 1 &= 0 & n \text{ is even} \\ x^{n-2} - 1 &= 0 & n \text{ is odd.} \end{aligned}$$

Clearly $\mathcal{G}_{h-1}^n = 0$ for $2 \leq h \leq n-2$ as it is the coefficient of x^{n-h-1} . Similarly $\mathcal{G}_{n-2}^n = 1$. Hence for computational basis vector v with Hamming weight h

$$\langle v|\mathcal{M}_n\rangle = \begin{cases} 0 & h = 0 \text{ by Eq. (12)} \\ (n-1)!(1/\sqrt{2})^{n-2} & h = 1 \text{ by Eq. (14)} \\ 0 & 2 \leq h \leq n-2 \text{ by Eq. (13)} \\ (n-1)!(1/\sqrt{2})^{n-2} & h = n-1 \text{ by Eq. (13)} \\ 0 & h = n \text{ by Eq. (12)} \end{cases} \quad (15)$$

Note that \tilde{W}_n is equivalently defined as the equal superposition of all computational basis vectors with Hamming weight equal to 1 and all computational basis vectors with Hamming weight equal to $n - 1$. Hence \mathcal{M}_n and \tilde{W}_n have identical computational basis decompositions up to a normalization factor

$$\mathcal{N} = 1/[\sqrt{2n}(n-1)!(1/\sqrt{2})^{n-2}]. \quad (16)$$

It follows immediately that $\mathcal{N} \cdot \mathcal{M}_n = \tilde{W}_n$ and hence Φ_n constitutes the Majorana representation of \tilde{W}_n , as claimed

Theorem 1. *W_n and \tilde{W}_n are SLOCC inequivalent for all $n > 2$.*

Proof. For $n > 2$, it follows from Lemma 1 that the Majorana representation of \tilde{W}_n contains $n - 2$ root terms \mathcal{U}_n constructed from members of \mathcal{R}_n . As each of the $n - 2$ terms of \mathcal{R}_n is distinct, each element \mathcal{U}_n is distinct and as each member of \mathcal{U}_n is trivially distinct from $|0\rangle$ and $|1\rangle$, each element of Φ_n is distinct. This implies a degeneracy configuration of $\underbrace{1, 1, \dots, 1}_n$

for \tilde{W}_n , which in turn implies the SLOCC inequivalence of the n -partite W and \tilde{W} [5]

3.2 SLOCC Relationship of \tilde{W} and GHZ

As W_3 and \tilde{W}_3 are SLOCC inequivalent, the 3-partite SLOCC classification [7] suggests that \tilde{W}_3 is SLOCC equivalent to GHZ_3 . Indeed, the invertible local operator (ILO) M_3 defined below transforms GHZ_3 to \tilde{W}_3 when applied symmetrically by each party. A computational search for $4 \leq n \leq 10$ yielded a unitary operator M_4 for the $n = 4$ case, but no ILOs for $5 \leq n \leq 10$. Hence \tilde{W}_4 and GHZ_4 are SLOCC equivalent, but it is unknown if this equivalence extends to larger n . The degeneracy configuration of GHZ_n is known to be $\underbrace{1, 1, \dots, 1}_n$ [5] and

so the argument used in Theorem 1 cannot be used to determine the SLOCC relationship of GHZ and \tilde{W} in the general case.

$$M_3 = \begin{pmatrix} \frac{e^{\frac{i\pi}{6}}}{\sqrt[3]{3}} & -\frac{e^{\frac{5i\pi}{6}}}{\sqrt[3]{3}} \\ -\frac{e^{\frac{5i\pi}{6}}}{\sqrt[3]{3}} & \frac{e^{\frac{i\pi}{6}}}{\sqrt[3]{3}} \end{pmatrix} \quad M_4 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-1+i}{2} \\ \frac{\sqrt{2}}{2} & \frac{1-i}{2} \end{pmatrix} \quad (17)$$

4 A TCALE Protocol using \tilde{W}

What remains is to construct a TCALE protocol on a quantum network. More precisely, I define the network model as in [2].

- There are n anonymous processors with a local classical state and a local quantum state. Processors are anonymous when each local classical state is initially identical and when the initial quantum state across the network is symmetric.
- Processors communicate classical information but *not* quantum information in synchronous rounds of faultless *broadcasting*. All messages within a round are sent to all other parties simultaneously.
- Each processor can perform local classical computation and local quantum computation.

Theorem 2. *There exists a TCALE protocol on the above network model for all $n > 2$, where the initial quantum state of the network is \tilde{W}_n .*

Proof. Consider the following protocol on a connected n party network with arbitrary topology, where $n > 2$. For each party i set their initial classical state $c = \text{null}$ and their initial quantum state q = the i th qubit of \tilde{W}_n .

```

1)  $c := \text{measure } q$ 
2) if  $c = 0$ 
    count_zeros := 1
    count_ones := 0
  else if  $c = 1$ 
    count_ones := 1
    count_zeros := 0
4) broadcast  $c$ 
5) wait until  $n-1$  messages are received
6) for all  $j$  in received messages
    if  $j = 1$  count_ones += 1
    if  $j = 0$  count_zeros += 1
7) if count_ones > count_zeros and  $c = 0$ 
    leader := true
  else if count_zeros > count_ones and  $c = 1$ 
    leader := true
  else
    leader := false

```

Here the command **measure** refers to a measurement in the computational basis.

By definition of \tilde{W}_n , after step 1 there will be a single processor i with unique measurement result $c_i \in \{0, 1\}$ such that for all $j \neq i$, $c_j = \overline{c_i}$. As broadcasting is faultless, after step 6 all processors will have an accurate count of the other parties' measurement results. Each processor uses this to correctly determine if their measurement result is unique. If it is, they mark themselves as the leader. As only processor i measures a unique result, only processor i terminates with **leader = true**, and so the protocol is correct. \tilde{W}_n is a superposition of permutations of computational basis terms and hence is symmetric, and so the protocol is anonymous. As the protocol is trivially guaranteed to terminate in $\mathcal{O}(n)$ steps, it is totally correct and anonymous as required \square .

It follows immediately from Theorem 1 and Theorem 2 that there exists a non- W -like state which enables a TCALE protocol. This proves that W -like states are not necessary for totally correct quantum anonymous leader election.

5 Conclusions

Preparing a n -partite W state is certainly non-trivial, and over the past decade much effort has been put into devising a variety of experimental preparation methods [10, 11, 12]. However the overhead of scalable preparation tends to be superpolynomial in the number of qubits [12], which is insufficient for any polynomial time algorithm that relies on the efficient creation of W_n . Furthermore, in order to be used for distributed computing the entanglement must be shared across the network without decoherence. Although there are proposed methods for creating W states from distant atoms [11], decoherence remains a major obstacle. As such any necessity of W states for distributed tasks could perhaps inhibit the feasibility of future distributed algorithms.

In this paper, I have provided a totally correct anonymous leader election protocol that uses a quantum resource that is potentially easier to create. In doing so I have corrected an important necessity result of distributed quantum computing, which furthers our understanding of how non-local entangled states can be used as a resource for distributed computing tasks. I summarize the corrections as follows.

The arguments of D’Hondt and Panangaden’s paper imply that in the case of pre-sharing a single qubit per processor to create a pure state, the state cannot allow k -symmetric paths for k different from 1 or $n - 1$. In actuality, this implies that states of the form $\alpha W_n + \beta \overline{W}_n$ ($|\alpha|^2 + |\beta|^2 = 1$) are necessary, not all of which are SLOCC equivalent to the W_n state. The protocol I present clearly provides TCALE for all such α and β , and so I conclude that $\alpha W_n + \beta \overline{W}_n$ ($|\alpha|^2 + |\beta|^2 = 1$) is the entire set of necessary and sufficient pure states for totally correct anonymous leader election in the single qubit per processor case.

In the case of multiple qubits per processor, the previous results are based on the assumption that the set of measurement results which will result in a leader or a follower is distinct and previously known by each party. The protocol presented in this paper shows that their assumption is incorrect. At the beginning of the protocol presented here, the measurement results of each processor cannot be split into groups of “leader” and “not leader”. Whether the leader will terminate with $c = 0$ or $c = 1$ is undecided at the start of the protocol, and the ambiguity is resolved by local quantum measurement and classical communication throughout the course of execution. This assumption is fundamental to D’Hondt and Panangaden’s definition of W -like states. As such their definition is too limiting, and their W -like states are in fact not necessary pure states for totally correct anonymous leader election.

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